9-3 Properties of Logarithms

Main Ideas

- Simplify and evaluate expressions using the properties of logarithms.
- Solve logarithmic equations using the properties of logarithms.

GET READY for the Lesson

In Lesson 6-1, you learned that the product of powers is the sum of their exponents.

$$9 \cdot 81 = 3^2 \cdot 3^4 \text{ or } 3^{2+4}$$

In Lesson 9-2, you learned that logarithms *are* exponents, so you might expect that a similar property applies to logarithms. Let's consider a specific case. Does $\log_3 (9 \cdot 81) = \log_3 9 + \log_3 81$? Investigate by simplifying the expression on each side of the equation.

$\log_3 (9 \cdot 81) = \log_3 (3^2 \cdot 3^4)$	Replace 9 with 3^2 and 81 with 3^4.
$= \log_3 3^{(2+4)}$	Product of Powers
$= 2 + 4$ or 6	Inverse Property of Exponents and Logarithms

$\log_3 9 + \log_3 81 = \log_3 3^2 + \log_3 3^4$	Replace 9 with 3^2 and 81 with 3^4.
$= 2 + 4$ or 6	Inverse Property of Exponents and Logarithms

Both expressions are equal to 6. So, $\log_3 (9 \cdot 81) = \log_3 9 + \log_3 81$.

Properties of Logarithms Since logarithms are exponents, the properties of logarithms can be derived from the properties of exponents. The Product Property of Logarithms can be derived from the Product of Powers Property of Exponents.

KEY CONCEPT — *Product Property of Logarithms*

Words The logarithm of a product is the sum of the logarithms of its factors.

Symbols For all positive numbers m, n, and b, where $b \neq 1$, $\log_b mn = \log_b m + \log_b n$.

Example $\log_3 (4)(7) = \log_3 4 + \log_3 7$

To show that this property is true, let $b^x = m$ and $b^y = n$. Then, using the definition of logarithm, $x = \log_b m$ and $y = \log_b n$.

$b^x b^y = mn$	Substitution
$b^{x+y} = mn$	Product of Powers
$\log_b b^{x+y} = \log_b mn$	Property of Equality for Logarithmic Functions
$x + y = \log_b mn$	Inverse Property of Exponents and Logarithms
$\log_b m + \log_b n = \log_b mn$	Replace x with $\log_b m$ and y with $\log_b n$.

You can use the Product Property of Logarithms to approximate logarithmic expressions.

Study Tip

Answer Check

You can check this answer by evaluating $2^{5.5850}$ on a calculator. The calculator should give a result of about 48, since $\log_2 48 \approx 5.5850$ means $2^{5.5850} \approx 48$.

EXAMPLE 1 Use the Product Property

Use $\log_2 3 \approx 1.5850$ to approximate the value of $\log_2 48$.

$\log_2 48 = \log_2 (2^4 \cdot 3)$	Replace 48 with $16 \cdot 3$ or $2^4 \cdot 3$.
$= \log_2 2^4 + \log_2 3$	Product Property
$= 4 + \log_2 3$	Inverse Property of Exponents and Logarithms
$\approx 4 + 1.5850$ or 5.5850	Replace $\log_2 3$ with 1.5850.

Thus, $\log_2 48$ is approximately 5.5850.

CHECK Your Progress

1. Use $\log_4 2 = 0.5$ to approximate the value of $\log_4 32$.

Recall that the quotient of powers is found by subtracting exponents. The property for the logarithm of a quotient is similar.

KEY CONCEPT — *Quotient Property of Logarithms*

Words The logarithm of a quotient is the difference of the logarithms of the numerator and the denominator.

Symbols For all positive numbers m, n, and b, where $b \neq 1$, $\log_b \frac{m}{n} = \log_b m - \log_b n$.

You will prove this property in Exercise 51.

EXAMPLE 2 Use the Quotient Property

Use $\log_3 5 \approx 1.4650$ and $\log_3 20 \approx 2.7268$ to approximate $\log_3 4$.

$\log_3 4 = \log_3 \frac{20}{5}$	Replace 4 with the quotient $\frac{20}{5}$.
$= \log_3 20 - \log_3 5$	Quotient Property
$\approx 2.7268 - 1.4650$ or 1.2618	$\log_3 20 \approx 2.7268$ and $\log_3 5 \approx 1.4650$

Thus, $\log_3 4$ is approximately 1.2618.

CHECK Use the definition of logarithm and a calculator.

3 [^] 1.2618 [ENTER] 3.999738507

Since $3^{1.2618} \approx 4$, the answer checks. ✓

CHECK Your Progress

2. Use $\log_5 7 \approx 1.2091$ and $\log_5 21 \approx 1.8917$ to approximate $\log_5 3$.

Real-World Career...

Sound Technician

Sound technicians produce movie sound tracks in motion picture production studios, control the sound of live events such as concerts, or record music in a recording studio.

Math Online

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Real-World EXAMPLE

3 **SOUND** **The loudness L of a sound is measured in decibels and is given by $L = 10 \log_{10} R$, where R is the sound's relative intensity. Suppose one person talks with a relative intensity of 10^6 or 60 decibels. Would the sound of ten people each talking at that same intensity be ten times as loud, or 600 decibels? Explain your reasoning.**

Let L_1 be the loudness of one person talking. → $L_1 = 10 \log_{10} 10^6$

Let L_2 be the loudness of ten people talking. → $L_2 = 10 \log_{10} (10 \cdot 10^6)$

Then the increase in loudness is $L_2 - L_1$.

$L_2 - L_1 = 10 \log_{10} (10 \cdot 10^6) - 10 \log_{10} 10^6$ Substitute for L_1 and L_2.

$= 10(\log_{10} 10 + \log_{10} 10^6) - 10 \log_{10} 10^6$ Product Property

$= 10 \log_{10} 10 + 10 \log_{10} 10^6 - 10 \log_{10} 10^6$ Distributive Property

$= 10 \log_{10} 10$ Subtract.

$= 10(1)$ or 10 Inverse Property of Exponents and Logarithms

The sound of ten people talking is perceived by the human ear to be only about 10 decibels louder than the sound of one person talking, or 70 decibels.

CHECK Your Progress

3. How much louder would 100 people talking at the same intensity be than just one person?

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Recall that the power of a power is found by multiplying exponents. The property for the logarithm of a power is similar.

KEY CONCEPT ***Power Property of Logarithms***

Words	The logarithm of a power is the product of the logarithm and the exponent.
Symbols	For any real number p and positive numbers m and b, where $b \neq 1$, $\log_b m^p = p \log_b m$.

You will prove this property in Exercise 45.

EXAMPLE Power Property of Logarithms

4 **Given $\log_4 6 \approx 1.2925$, approximate the value of $\log_4 36$.**

$\log_4 36 = \log_4 6^2$ Replace 36 with 6^2.

$= 2 \log_4 6$ Power Property

$\approx 2(1.2925)$ or 2.585 Replace $\log_4 6$ with 1.2925.

CHECK Your Progress

4. Given $\log_3 7 \approx 1.7712$, approximate the value of $\log_3 49$.

Solve Logarithmic Equations

Solve Logarithmic Equations You can use the properties of logarithms to solve equations involving logarithms.

EXAMPLE 5 Solve Equations Using Properties of Logarithms

Solve each equation.

a. $3 \log_5 x - \log_5 4 = \log_5 16$

$3 \log_5 x - \log_5 4 = \log_5 16$	Original equation
$\log_5 x^3 - \log_5 4 = \log_5 16$	Power Property
$\log_5 \frac{x^3}{4} = \log_5 16$	Quotient Property
$\frac{x^3}{4} = 16$	Property of Equality for Logarithmic Functions
$x^3 = 64$	Multiply each side by 4.
$x = 4$	Take the cube root of each side.

The solution is 4.

b. $\log_4 x + \log_4 (x - 6) = 2$

$\log_4 x + \log_4 (x - 6) = 2$	Original equation
$\log_4 x(x - 6) = 2$	Product Property
$x(x - 6) = 4^2$	Definition of logarithm
$x^2 - 6x - 16 = 0$	Subtract 16 from each side.
$(x - 8)(x + 2) = 0$	Factor.
$x - 8 = 0$ or $x + 2 = 0$	Zero Product Property
$x = 8$ $\quad x = -2$	Solve each equation.

CHECK Substitute each value into the original equation.

$\log_4 8 + \log_4 (8 - 6) \stackrel{?}{=} 2$

$\log_4 8 + \log_4 2 \stackrel{?}{=} 2$

$\log_4 (8 \cdot 2) \stackrel{?}{=} 2$

$\log_4 16 \stackrel{?}{=} 2$

$2 = 2$ ✓

$\log_4 (-2) + \log_4 (-2 - 6) \stackrel{?}{=} 2$

$\log_4 (-2) + \log_4 (-8) \stackrel{?}{=} 2$

Since $\log_4 (-2)$ and $\log_4 (-8)$ are undefined, -2 is an extraneous solution and must be eliminated.

The only solution is 8.

Study Tip

Checking Solutions

It is wise to check all solutions to see if they are valid since the domain of a logarithmic function is not the complete set of real numbers.

CHECK Your Progress

5A. $2 \log_7 x = \log_7 27 + \log_7 3$ **5B.** $\log_6 x + \log_6 (x + 5) = 2$

CHECK Your Understanding

Examples 1, 2 (p. 521)

Use $\log_3 2 \approx 0.6309$ and $\log_3 7 \approx 1.7712$ to approximate the value of each expression.

1. $\log_3 18$ **2.** $\log_3 14$ **3.** $\log_3 \frac{7}{2}$ **4.** $\log_3 \frac{2}{3}$

Example 3 (p. 522)

5. **MOUNTAIN CLIMBING** As elevation increases, the atmospheric air pressure decreases. The formula for pressure based on elevation is $a = 15{,}500\,(5 - \log_{10} P)$, where a is the altitude in meters and P is the pressure in pascals (1 psi ≈ 6900 pascals). What is the air pressure at the summit in pascals for each mountain listed in the table at the right?

Mountain	Country	Height (m)
Everest	Nepal/Tibet	8850
Trisuli	India	7074
Bonete	Argentina/Chile	6872
McKinley	United States	6194
Logan	Canada	5959

Source: infoplease.com

Example 4 (p. 522)

Given $\log_2 7 \approx 2.8074$ and $\log_5 8 \approx 1.2920$ to approximate the value of each expression.

6. $\log_2 49$
7. $\log_5 64$

Example 5 (p. 523)

Solve each equation. Check your solutions.

8. $\log_3 42 - \log_3 n = \log_3 7$
9. $\log_2(3x) + \log_2 5 = \log_2 30$
10. $2 \log_5 x = \log_5 9$
11. $\log_{10} a + \log_{10} (a + 21) = 2$

Exercises

HOMEWORK HELP	
For Exercises	See Examples
12–14	1
15–17	2
18–20	3
21–24	4
25–30	5

Use $\log_5 2 \approx 0.4307$ and $\log_5 3 \approx 0.6826$ to approximate the value of each expression.

12. $\log_5 50$
13. $\log_5 30$
14. $\log_5 20$
15. $\log_5 \frac{2}{3}$
16. $\log_5 \frac{3}{2}$
17. $\log_5 \frac{4}{3}$
18. $\log_5 9$
19. $\log_5 8$
20. $\log_5 16$

21. **EARTHQUAKES** The great Alaskan earthquake, in 1964, was about 100 times as intense as the Loma Prieta earthquake in San Francisco, in 1989. Find the difference in the Richter scale magnitudes of the earthquakes.

PROBABILITY **For Exercises 22–24, use the following information.**
In the 1930s, Dr. Frank Benford demonstrated a way to determine whether a set of numbers have been randomly chosen or the numbers have been manually chosen. If the sets of numbers were not randomly chosen, then the Benford formula, $P = \log_{10}\left(1 + \frac{1}{d}\right)$, predicts the probability of a digit d being the first digit of the set. For example, there is a 4.6% probability that the first digit is 9.

22. Rewrite the formula to solve for the digit if given the probability.
23. Find the digit that has a 9.7% probability of being selected.
24. Find the probability that the first digit is 1 ($\log_{10} 2 \approx 0.30103$).

Solve each equation. Check your solutions.

25. $\log_3 5 + \log_3 x = \log_3 10$
26. $\log_4 a + \log_4 9 = \log_4 27$
27. $\log_{10} 16 - \log_{10} (2t) = \log_{10} 2$
28. $\log_7 24 - \log_7 (y + 5) = \log_7 8$
29. $\log_2 n = \frac{1}{4} \log_2 16 + \frac{1}{2} \log_2 49$
30. $2 \log_{10} 6 - \frac{1}{3} \log_{10} 27 = \log_{10} x$

Solve for n.

31. $\log_a (4n) - 2\log_a x = \log_a x$

32. $\log_b 8 + 3\log_b n = 3\log_b (x - 1)$

Solve each equation. Check your solutions.

33. $\log_{10} z + \log_{10} (z + 3) = 1$

34. $\log_6 (a^2 + 2) + \log_6 2 = 2$

35. $\log_2 (12b - 21) - \log_2 (b^2 - 3) = 2$

36. $\log_2 (y + 2) - \log_2 (y - 2) = 1$

37. $\log_3 0.1 + 2\log_3 x = \log_3 2 + \log_3 5$

38. $\log_5 64 - \log_5 \frac{8}{3} + \log_5 2 = \log_5 (4p)$

Real-World Link

The Greek astronomer Hipparchus made the first known catalog of stars. He listed the brightness of each star on a scale of 1 to 6, the brightest being 1. With no telescope, he could only see stars as dim as the 6th magnitude.

Source: NASA

SOUND **For Exercises 39–41, use the formula for the loudness of sound in Example 3 on page 546. Use $\log_{10} 2 \approx 0.3010$ and $\log_{10} 3 \approx 0.4771$.**

39. A certain sound has a relative intensity of R. By how many decibels does the sound increase when the intensity is doubled?

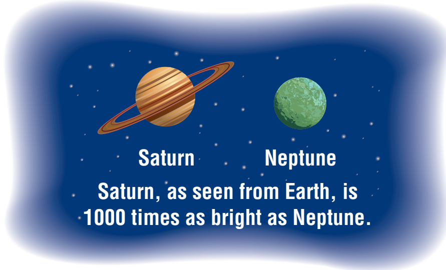
40. A certain sound has a relative intensity of R. By how many decibels does the sound decrease when the intensity is halved?

41. A stadium containing 10,000 cheering people can produce a crowd noise of about 90 decibels. If everyone cheers with the same relative intensity, how much noise, in decibels, is a crowd of 30,000 people capable of producing? Explain your reasoning.

STAR LIGHT **For Exercises 42–44, use the following information.**

The brightness, or apparent magnitude, m of a star or planet is given by $m = 6 - 2.5\log_{10} \frac{L}{L_0}$, where L is the amount of light L coming to Earth from the star or planet and L_0 is the amount of light from a sixth magnitude star.

The crescent moon is about 100 times as bright as the brightest star, Sirius.



Saturn, as seen from Earth, is 1000 times as bright as Neptune.

42. Find the difference in the magnitudes of Sirius and the crescent moon.

43. Find the difference in the magnitudes of Saturn and Neptune.

44. **RESEARCH** Use the Internet or other reference to find the magnitude of the dimmest stars that we can now see with ground-based telescopes.

EXTRA PRACTICE
See pages 910, 934.

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H.O.T. Problems

45. **REASONING** Use the properties of exponents to prove the Power Property of Logarithms.

46. **REASONING** Use the properties of Logarithms to prove that $\log_a \frac{1}{x} = -\log_a x$.

47. **CHALLENGE** Simplify $\log_{\sqrt{a}} (a^2)$ to find an exact numerical value.

48. **CHALLENGE** Simplify $x^{3\log_x 2 - \log_x 5}$ to find an exact numerical value.

CHALLENGE **Tell whether each statement is *true* or *false*. If true, show that it is true. If false, give a counterexample.**

49. For all positive numbers m, n, and b, where $b \neq 1$, $\log_b (m + n) = \log_b m + \log_b n$.

50. For all positive numbers m, n, x, and b, where $b \neq 1$, $n \log_b x + m \log_b x = (n + m) \log_b x$.

51. **REASONING** Use the properties of exponents to prove the Quotient Property of Logarithms.

52. ***Writing in Math*** Use the information given regarding exponents and logarithms on page 520 to explain how the properties of exponents and logarithms are related. Include examples like the one shown at the beginning of the lesson illustrating the Quotient Property and Power Property of Logarithms, and an explanation of the similarity between one property of exponents and its related property of logarithms in your answer.

STANDARDIZED TEST PRACTICE

53. **ACT/SAT** To what is $2 \log_5 12 - \log_5 8 - 2 \log_5 3$ equal?

A $\log_5 2$

B $\log_5 3$

C $\log_5 0.5$

D 1

54. **REVIEW** In a movie theater, 2 boys and 3 girls are seated randomly together. What is the probability that the 2 boys are seated next to each other?

F $\frac{1}{5}$ G $\frac{2}{5}$ H $\frac{1}{2}$ J $\frac{2}{3}$

Spiral Review

Evaluate each expression. (Lesson 9-2)

55. $\log_3 81$

56. $\log_9 \frac{1}{729}$

57. $\log_7 7^{2x}$

Solve each equation or inequality. Check your solutions. (Lesson 9-1)

58. $3^{5n + 3} = 3^{33}$

59. $7^a = 49^{-4}$

60. $3^{d + 4} > 9^d$

61. **PHYSICS** If a stone is dropped from a cliff, the equation $t = \frac{1}{4}\sqrt{d}$ represents the time t in seconds that it takes for the stone to reach the ground. If d represents the distance in feet that the stone falls, find how long it would take for a stone to hit the ground after falling from a 150-foot cliff. (Lesson 7-2)

GET READY for the Next Lesson

PREREQUISITE SKILL **Solve each equation or inequality. Check your solutions.** (Lesson 9-2)

62. $\log_3 x = \log_3 (2x - 1)$

63. $\log_{10} 2^x = \log_{10} 32$

64. $\log_2 3x > \log_2 5$

65. $\log_5 (4x + 3) < \log_5 11$